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Chiral oscillations in terms of the zitterbewegung effect

A.E. Bernardini[a]

Instituto de Física Gleb Wataghin, UNICAMP, PO Box 6165, 13083-970, Campinas, SP, Brasil

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Abstract. We seek the *immediate* description of chiral oscillations in terms of the trembling motion described by the velocity (Dirac) operator $\boldsymbol{\alpha}$. By taking into account the complete set of Dirac equation solutions, which results in a free propagating Dirac wave packet composed by positive and negative frequency components, we report about the well-established zitterbewegung results and indicate how chiral oscillations can be expressed in terms of the well-known quantum oscillating variables. We conclude with the interpretation of chiral oscillations as very rapid position oscillation projections onto the longitudinally decomposed direction of the motion.

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When Dirac had introduced his fundamental equation for the relativistic quantum mechanics of spin-half particles [1, 2], it became soon clear that in spite of its overwhelming success in describing very accurately the energy levels of the hydrogen atoms, this theory was replete of *apparent* inconsistencies like the presence of negative frequency solutions, the Klein paradox [3] and the zitterbewegung effect [4]. To be more specific, in first quantization, the relativistic Dirac equation predicts the existence for the spin-half particle, under certain conditions, of an oscillating time dependence in the average of the position variable. This phenomenon, which is known as the zitterbewegung, was formerly noticed by Schroedinger [4] as a consequence of the non-commutative relation between the position operator $\mathbf{x}$ and the Hamiltonian H. The existence of the zitterbewegung effect is not stranger than the existence of negative frequency solutions, since, in fact, such a trembling motion is only manifest for wave functions with significant interference between positive and negative frequency solutions of the Dirac equation. In atomic physics, the electron exhibits these violent quantum fluctuations in the position and becomes sensitive to an effective potential, which could explain the Darwin term in the hydrogen atom [5].[1] In the recent literature, we have witnessed some interest in identifying the physical meaning of the variables that coexist with the interference between positive and negative frequency solutions of the Dirac equation [6–9]. In this context, by following an almost identical line of reasoning as that applied for identifying rapid oscillations of the position, it is possible to verify that the expectation value of the Dirac chiral operator γ^5 also presents a peculiar oscillatory behavior. In fact, the formalism with Dirac wave packets [5, 10] leads to the study of chiral oscillations [11]. In the standard model flavor-changing interactions, neutrinos with positive chirality are decoupled from the neutrino absorbing charged weak currents [11], so that chirality coupled to flavor oscillations could lead to some small modifications to the standard flavor conversion formula [12]. Due to this tenuous relation between zitterbewegung and chiral oscillations, the question we shall answer in this article is related to the *immediate* description of chiral oscillations in terms of the zitterbewegung motion, i.e. we will demonstrate that chiral oscillations are coupled with the zitterbewegung motion in such a way that they cannot exist independently of each other. Moreover, if we observe that very rapid oscillations of the position variable can be decomposed into transversal and longitudinal polarization vector directions, we can also interpret chiral oscillations as very rapid position oscillation projections onto the momentum direction. The answer to the posed question is obtained by means of very simple mathematical manipulations; nevertheless, we believe that, in spite of its simplicity, it is important in the larger context of quantum oscillation phenomena.

All the following deductions concern a free propagating *fermionic* particle; thus, let us consider the covariant free particle Dirac equation given by

$$(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0\,, \tag{1}$$

[a] e-mail: alexeb@ifi.unicamp.br

[1] One can point out the possibility of avoiding zitterbewegung by redefining the spatial variable via a Foldy–Wouthuysen transformation, which introduces a very complicated position variable. Anyway, the coexistence of negative and positive Dirac equation solutions (particles and anti-particles) can be rigorously realized only in terms of a quantum field theory prescription.

where $x=(t,\mathbf{x})$, and we have used the usual representation found in the literature [5, 13, 14] with $c=\hbar=1$. The plane wave solutions are expressed in terms of $\psi(x)=\psi_+(x)+\psi_-(x)$, with

$$\begin{aligned}\psi_+(x)&=\mathrm{e}^{[-\mathrm{i}px]}u(p) \quad \text{for positive frequencies}\,,\\ \psi_-(x)&=\mathrm{e}^{[+\mathrm{i}px]}v(p) \quad \text{for negative frequencies}\,, \end{aligned}\tag{2}$$

where p is the relativistic four-momentum, $p=(E,\mathbf{p})$ with $E^2=m^2+\mathbf{p}^2$, and the free propagating mass eigenstate spinors are given by [16]

$$\begin{aligned}u_s(p)&=\frac{\gamma^\mu p_\mu+m}{[2E(m+E)]^{1/2}}u_s(m,0)=\begin{pmatrix}\left(\frac{E+m}{2E}\right)^{1/2}\eta_s\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{[2E(E+m)]^{1/2}}\eta_s\end{pmatrix},\\ v_s(p)&=\frac{-\gamma^\mu p_\mu+m}{[2E(m+E)]^{1/2}}v_s(m,0)=\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{[2E(E+m)]^{1/2}}\eta_s\\ \left(\frac{E+m}{2E}\right)^{1/2}\eta_s\end{pmatrix},\end{aligned}\tag{3}$$

with $\overline{u}(p)$ (or $\overline{v}(p)$) defined by $\overline{u}(p)=u^\dagger(p)\gamma^0$ (or $\overline{v}(p)=v^\dagger(p)\gamma^0$). We could also resort to the Dirac wave packet formalism [13], where the general procedure consists of writing the general (complete) Dirac equation (wave packet) solution of (2) as

$$\begin{aligned}\psi(t,\mathbf{x})=\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\sum_{s=1,2}&\{b_s(p)u_s(p)\exp[-\mathrm{i}Et]\\ &+d_s^*(\tilde{p})v_s(\tilde{p})\exp[+\mathrm{i}Et]\}\exp[\mathrm{i}\mathbf{p}\cdot\mathbf{x}]\,,\end{aligned}\tag{4}$$

where $\overline{u}(p)$ (or $\overline{v}(p)$) is defined by $\overline{u}(p)=u^\dagger(x)\gamma^0$ (or $\overline{v}(p)=v^\dagger(p)\gamma^0$) and $\tilde{p}=(E,-\mathbf{p})$. By fixing the initial condition over $\psi(0,\mathbf{x})$ as the Fourier transform of the weight function

$$\varphi(\mathbf{p}-\mathbf{p}_0)w=\sum_{s=1,2}\{b_s(p)u_s(p)+d_s^*(\tilde{p})v_s(\tilde{p})\}\,,\tag{5}$$

we get

$$\psi(0,\mathbf{x})=\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\varphi(\mathbf{p}-\mathbf{p}_0)\exp[\mathrm{i}\mathbf{p}\cdot\mathbf{x}]w\,,\tag{6}$$

where w is some fixed normalized spinor. The coefficients $b_s(p)$ and $d_s^*(\tilde{p})$ can thus be calculated by using the orthogonality properties of Dirac spinors. These coefficients carry important physical information. For *any* initial state $\psi(0,\mathbf{x})$ given by (6), the negative frequency solution coefficient $d_s^*(\tilde{p})$ necessarily provides a non-zero contribution to the time evolving wave packet. This forces us to take the complete set of Dirac equation solutions to construct the wave packet. Only if we consider a momentum distribution given by a delta function (plane wave limit) and suppose the initial spinor w to be a positive energy mass eigenstate with momentum $\mathbf{p}$, the contribution due to the negative frequency solutions $d_s^*(\tilde{p})$ will become zero.

By writing the free propagating particle Dirac Hamiltonian as

$$H=\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m\,,\tag{7}$$

where $\boldsymbol{\alpha}=\sum_{k=1}^3\alpha_k\hat{k}=\sum_{k=1}^3\gamma_0\gamma_k\hat{k}$ and $\beta=\gamma_0$, we can easily verify whether a given observable $\mathcal{O}$ is a constant of motion, by means of the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathcal{O}\rangle=\mathrm{i}\langle[H,\mathcal{O}]\rangle+\left\langle\frac{\partial\mathcal{O}}{\partial t}\right\rangle.\tag{8}$$

Equation (8) aids us in verifying that, for instance, the free propagating particle momentum is a conserved quantity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{p}\rangle=\mathrm{i}\langle[H,\mathbf{p}]\rangle=0\,.\tag{9}$$

Otherwise, the particle velocity given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{x}\rangle=\mathrm{i}\langle[H,\mathbf{x}]\rangle=\langle\boldsymbol{\alpha}\rangle\tag{10}$$

turns out to have a non-zero value. At first sight this seems to be quite reasonable [13] when we calculate the averaged value of the positive frequency solution ($\psi_+(x)$):

$$\begin{aligned}\langle\boldsymbol{\alpha}\rangle_+(t)&=\int\mathrm{d}^3\mathbf{x}\psi_+^\dagger(x)\boldsymbol{\alpha}\psi_+(x)\\ &=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathbf{p}}{E}\sum_{s=1,2}|b_s(p)|^2\,,\end{aligned}\tag{11}$$

which represents exactly the expectation value of the relativistic velocity $\frac{\mathbf{p}}{E}$. Note, however, that the eigenvalue of α_k is ±1 and corresponds to $\pm c$, but we know that, for massive particles, the classical velocity cannot be equal to $\pm c$. Moreover, once α_k and α_l do not commute when $k\neq l$, the measurement of the x-component of the velocity is incompatible with the measurement of the y-component of the velocity. This may seem to be unusual, since we know that p_x and p_y commute. In spite of these peculiarities, there is no contradiction with the above results. The plane wave solutions, see (3), which are momentum and energy eigenfunctions, are not eigenfunctions of α_k. Anyway, the velocity related to $\boldsymbol{\alpha}(t)$ does not represent a constant of motion, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\alpha}\rangle=\mathrm{i}\langle[H,\boldsymbol{\alpha}(t)]\rangle=2\mathrm{i}\left(\langle\mathbf{p}\rangle-\langle\boldsymbol{\alpha}H\rangle\right)\,,\tag{12}$$

which can be regarded as a differential equation for $\boldsymbol{\alpha}(t)$. Keeping in mind that $\mathbf{p}$ and H are constants of motion, we can easily solve (12) in order to obtain [13]

$$\langle\boldsymbol{\alpha}\rangle(t)=\left\langle\mathbf{p}H^{-1}\right\rangle+\left\langle\left(\boldsymbol{\alpha}(0)-\mathbf{p}H^{-1}\right)\mathrm{e}^{[-2\mathrm{i}Ht]}\right\rangle.\tag{13}$$

By following the same procedure, we observe that the spin angular momentum of a free Dirac particle, represented by the operator $\boldsymbol{\Sigma}=\gamma^5\boldsymbol{\alpha}$, is not a constant of motion, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\Sigma}\rangle=\mathrm{i}\langle[H,\boldsymbol{\Sigma}]\rangle=-2(\langle\boldsymbol{\alpha}\rangle\times\mathbf{p})\,.\tag{14}$$

With the aid of the above result, it can easily be demonstrated that the particle *helicity* $h=\frac{1}{2}\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$, given by the projection of the spin angular momentum onto the momentum direction, also is a constant of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle h\rangle = \mathrm{i}\langle[H,h]\rangle = -\langle(\boldsymbol{\alpha}\times\mathbf{p})\cdot\hat{\mathbf{p}}\rangle = 0\,. \qquad (15)$$

At the same time, the chiral operator γ^5, of which very often the meaning is confused with the helicity operator h, is *not* a constant of motion [11], since

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma^5\rangle = \mathrm{i}\left\langle\left[H,\gamma^5\right]\right\rangle = 2\mathrm{i}m\left\langle\gamma^0\gamma^5\right\rangle\,. \qquad (16)$$

The effective value of (16) appears only when both positive and negative frequencies are taken into account to compose a Dirac wave packet; i.e., the non-zero mean value of $\langle\gamma_0\gamma_5\rangle$ is revealed by the interference between Dirac equation solutions with opposite sign frequencies. It is important to note that an eigenstate of helicity can be read as an eigenstate of chirality only in the ultra-relativistic limit (mass $m=0$) [16]. At time $t=0$, the coefficients $b_s(p)$ and $d^*_s(p)$ used in the construction of the Dirac wave packet $\psi(x)$ can be chosen to provide a negative (positive) chirality eigenstate [12, 15] or, in the same way, to provide a helicity eigenstate (when $hu_s(p)(v_s(p)) \equiv \pm\frac{1}{2}u_s(p)(v_s(p))$). Once we have assumed that the initial chiral eigenstate[2] $\psi(0,\mathbf{x})$ is not only a superposition of momentum eigenstates weighted by the momentum distribution of (5) centered around $\vec{\mathbf{p}_0}$, but also a helicity (constant of motion) eigenstate obtained through the production of a spin-polarized particle, which formally occurs when we assume that the constant spinor w in the wave packet expression (6) is a simultaneous eigenspinor of γ^5 and h,[3] then we can make use of the following decomposition:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma_5\rangle(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,(\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}})\rangle(t)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\rangle(t)\right)\langle\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\rangle(t)\\ &\quad+\langle\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\rangle(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\rangle(t)\right)\,, \qquad (17)\end{aligned}$$

from which, when we substitute (9) and (16), we can establish a subtle relation between the chirality operator $\gamma_5(t)$ and the velocity operator $\boldsymbol{\alpha}(t)$, in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma_5\rangle(t) = (2h)\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\alpha}\rangle(t)\right)\cdot\hat{\mathbf{p}}\,. \qquad (18)$$

The time evolution of γ_5 presents an oscillating character, which can be interpreted as a direct and natural consequence of the oscillating character of the trembling motion described by $\boldsymbol{\alpha}(t)$. The coupling of (12) and (18) leads to the explicit dependence of $\gamma_5(t)$ on the momentum variable, $\boldsymbol{\alpha}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma_5\rangle(t) = 4\mathrm{i}h\,(\mathbf{p}-\langle\boldsymbol{\alpha}(t)H\rangle)\cdot\hat{\mathbf{p}}\,. \qquad (19)$$

Since $\langle h\rangle$, $\mathbf{p}$ and H are constants of motion, we conclude that there will not be chiral oscillations without the "quivering motion" of the position. The mutual constraint for the chirality operator $\gamma_5(t)$ and the velocity operator $\boldsymbol{\alpha}(t)$ becomes more interesting when we take into account the complete expression for the current density $\overline{\psi}(x)\gamma_\mu\psi(x)$ (which leads to the averaged value of $\boldsymbol{\alpha}(t)$) and apply the Gordon decomposition [16],

$$\begin{aligned}\overline{\psi}(x)\gamma_\mu\psi(x) &= -\frac{\mathrm{i}}{2m}\left[\left(\partial^\mu\overline{\psi}(x)\right)\psi(x)-\overline{\psi}(x)\left(\partial^\mu\psi(x)\right)\right]\\ &\quad+\frac{1}{2m}\partial^\nu\left(\overline{\psi}(x)\sigma_{\mu\nu}\psi(x)\right)\,, \qquad (20)\end{aligned}$$

where $\sigma_{\mu\nu}=\frac{\mathrm{i}}{2}[\gamma_\mu,\gamma_\nu]$. The spatial integration of the vector components of (20) gives us

$$\begin{aligned}\int\mathrm{d}^3\mathbf{x}\psi^\dagger\boldsymbol{\alpha}\psi &= \frac{1}{2m}\int\mathrm{d}^3\mathbf{x}\left\{-\mathrm{i}\left[\overline{\psi}\left(\boldsymbol{\nabla}\psi\right)-\left(\boldsymbol{\nabla}\overline{\psi}\right)\psi\right]\right.\\ &\quad\left.+\left[\boldsymbol{\nabla}\times\left(\overline{\psi}\boldsymbol{\Sigma}\psi\right)-\mathrm{i}\partial_t\left(\overline{\psi}\boldsymbol{\alpha}\psi\right)\right]\right\}\,, \qquad (21)\end{aligned}$$

where we have suppressed the x dependence. By using the Dirac equation solution expressed by (4), we can write the decomposed components of $\langle\boldsymbol{\alpha}\rangle$ as

$$\int\mathrm{d}^3\mathbf{x}\boldsymbol{\nabla}\times\left(\overline{\psi}\boldsymbol{\Sigma}\psi\right) = 0\,, \qquad (22)$$

$$\begin{aligned}&\frac{\mathrm{i}}{2m}\int\mathrm{d}^3\mathbf{x}\left[\overline{\psi}\left(\boldsymbol{\nabla}\psi\right)-\left(\boldsymbol{\nabla}\overline{\psi}\right)\psi\right]\\ &=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{\frac{\mathbf{p}}{E}\sum_{s=1,2}\left[|b_s(p)|^2+|d_s(p)|^2\right]\right.\\ &\quad+\sum_{s=1,2}\left(\frac{m}{E}-\frac{E}{m}\right)a_s\hat{\boldsymbol{p}}\left[b^*_s(p)d^*_s(\tilde{p})\mathrm{e}^{[+2\mathrm{i}Et]}\right.\\ &\qquad\left.\left.-d_s(p)b_s(\tilde{p})\mathrm{e}^{[-2\mathrm{i}Et]}\right]\right\}\,, \qquad (23)\end{aligned}$$

[2] Neutrinos are supposedly produced as chiral eigenstates via weak interactions.

[3] When we establish that $\psi(0,\mathbf{x})$ is a h and/or a γ^5 eigenstate, we are referring to the choice of the fixed spinor w in (6). As a peculiarity, *breaking* of the Lorentz symmetry is not specifically related to the choice of w, but more generically to the choice of the momentum distributions $b_s(p)$ and $d^*_s(\tilde{p})$ written in terms of w and $\varphi(\mathbf{p}-\mathbf{p}_0)$. It is not the choice of w as a h eigenstate that ruins the Lorentz invariance of Dirac wave packets, but the general choice of the momentum distribution for constructing them and the effective way that they (the Dirac wave packets) appear in some averaged value integrals; i.e. once one has established an analytical shape for the momentum distribution (as has been done in [11] and [5]), it is stated that it is valid only for one specific reference frame, and the discussion of Lorentz invariance aspects becomes a little more complicated since it touches more fundamental definitions.

with $\tilde{p}=(E,-\mathbf{p})$, where we have assumed $a_s=\eta_{s'}\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\eta_s=(-1)^{s+1}\delta_{ss'}$, and

$$-\frac{\mathrm{i}}{2m}\int\mathrm{d}^3\mathbf{x}\partial_t\left(\overline{\psi}\boldsymbol{\alpha}\psi\right)$$
$$=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{\sum_{s=1,2}\left(\frac{E}{m}\right)a_s\hat{\mathbf{p}}\left[b_s^*(p)d_s^*(\tilde{p})\mathrm{e}^{[+2\mathrm{i}Et]}-d_s(p)b_s(\tilde{p})\mathrm{e}^{[-2\mathrm{i}Et]}\right]\right.$$
$$\left.+\sum_{s\neq s'}\hat{\mathbf{n}}_s\left[b_s^*(p)d_{s'}^*(\tilde{p})\mathrm{e}^{[+2\mathrm{i}Et]}-d_s(p)b_{s'}(\tilde{p})\mathrm{e}^{[-2\mathrm{i}Et]}\right]\right\},\qquad(24)$$

where $\hat{\mathbf{n}}_{1(2)}=\hat{1}\pm\mathrm{i}\hat{2}$ when $\hat{\mathbf{p}}=\hat{3}$ and the unitary vectors $\hat{1}$, $\hat{2}$ and $\hat{3}$ correspond to mutually orthogonal directions.

Equation (22) allows us to verify that the zitterbewegung does not get a contribution from the intrinsic spin dependent ($\boldsymbol{\Sigma}$) magnetic moment component, which couples with external magnetic fields $\mathbf{B}(x)$. In fact, the zitterbewegung originates from the current strictly related to the internal electric moment. To clear up this point, it is convenient to consider the modern and more precise interpretation of the zitterbewegung [17, 18]. From such a theoretical perspective, the zitterbewegung for a Dirac particle is clearly related to the separation between the center of mass, which is related to the Foldy–Wouthuysen position operator, and the center of charge, which corresponds to Dirac's position operator $\mathbf{x}$. It is the particle's charge at position $\mathbf{x}$ that is moving at the speed of light in circles of radius $\hbar/2mc$ around the center of mass, such that the average value of this velocity is related to the linear momentum of the particle. It is this separation, and thus the existence of an electric dipole moment with respect to the center of mass, that justifies the above statement. For both particles and antiparticles, now in pure positive or negative energy states, this internal motion of the charge around the center of mass exists, but there are no chiral oscillations, which is no longer obvious. In the wave packet construction, it should be equivalent to assume only positive (or only negative) Dirac equation frequency solutions (like in (11)) in the calculation of (observable) averaged values, in particular, for the γ^5 time-derivative operator (which would lead to $\mathrm{d}\left\langle\gamma^5\right\rangle/\mathrm{d}t=0$). However, the production of a fermionic particle as a chiral eigenstate (for instance, a neutrino), imposes the wave packet space-time evolution described by the superposition of positive and negative frequency solutions of the Dirac equation [10]. It recovers the possibility of chiral oscillations for massive fermionic particles. In this context, the coupling between chiral oscillations and the zitterbewegung is recovered.

By taking into account the (momentum direction components of the) equations (23)–(24), we can turn back to (19), which carries the main idea of this manuscript, and observe that chiral oscillations can be essentially constructed in terms of the longitudinal components of $\langle\boldsymbol{\alpha}\rangle$. By calculating the mean value of $\langle\boldsymbol{\alpha}H\rangle$ and projecting it onto the momentum direction $\hat{\mathbf{p}}$, we obtain

$$\langle\boldsymbol{\alpha}(t)H\rangle\cdot\hat{\mathbf{p}}$$
$$=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{\frac{\mathbf{p}}{E}\cdot\hat{\boldsymbol{p}}\sum_{s=1,2}\left[(E)|b_s(p)|^2+(E)|d_s(p)|^2\right]\right.$$
$$+\sum_{s=1,2}\frac{m}{E}a_s\left[(E)b_s^*(p)d_s^*(\tilde{p})\mathrm{e}^{[+2\mathrm{i}Et]}\right.$$
$$\left.\left.-(E)d_s(p)b_s(\tilde{p})\mathrm{e}^{[-2\mathrm{i}Et]}\right]\right\}$$
$$=|\mathbf{p}|-\int\frac{\mathrm{d}^3p}{(2\pi)^3}m\sum_{s=1,2}a_s\left[b_s^*(p)d_s^*(\tilde{p})\mathrm{e}^{[+2\mathrm{i}Et]}-d_s(p)b_s(\tilde{p})\mathrm{e}^{[-2\mathrm{i}Et]}\right],\qquad(25)$$

which can be substituted in (19) in order to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma_5\rangle(t)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m}{E}\sum_{s=1,2}(2ha_s)(2\mathrm{i}E)\times\left[b_s^*(p)d_s^*(\tilde{p})\mathrm{e}^{[+2\mathrm{i}Et]}-d_s(p)b_s(\tilde{p})\mathrm{e}^{[-2\mathrm{i}Et]}\right];\qquad(26)$$

therefore, the time evolution of the chirality operator could be written as

$$\langle\gamma_5\rangle(t)=\langle\gamma_5\rangle(0)+\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m}{E}\times\sum_{s=1,2}(2ha_s)\left[d_s(p)b_s(\tilde{p})\left(\mathrm{e}^{[2\mathrm{i}Et]}-1\right)+\mathrm{h.c.}\right]$$
$$=\langle\gamma_5\rangle(0)+\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m}{E}\times\sum_{s=1,2}\left[d_s(p)b_s(\tilde{p})\left(\mathrm{e}^{[2\mathrm{i}Et]}-1\right)+\mathrm{h.c.}\right],\qquad(27)$$

since $2ha_s$ is equal to the unity for well defined spin up/down (helicity) states.

The physical significance of the above results is discussed in a very subtle way in [11]. The authors of this reference demonstrated that, for a gaussian initial wave function and if the initial state has average chirality zero, the oscillations of the *left–right* (L–R) chiralities cancel, and there is again no overall oscillation. This could be the origin of an apparent paradox. In order to reproduce the ideas presented in [11], we observe that for any mass eigenstate represented by plane wave solutions of the Dirac equation with mass, the rest-frame wave function is always an equal mixture of both chiralities. This is easily verified when

$$\psi=\frac{1-\gamma^5}{2}\psi+\frac{1+\gamma^5}{2}\psi=\psi_{\mathrm{L}}+\psi_{\mathrm{R}},\qquad(28)$$

where the $\psi_{\mathrm{L,R}}$ correspond respectively to the chirality quantum numbers ∓1. It could be shown that, in the rest frame of a particle, we have

$$|\psi_{\mathrm{L}}|^2=|\psi_{\mathrm{R}}|^2.\qquad(29)$$

Note that this result is not Lorentz invariant, since a Lorentz *boost* is not a unitary transformation. Thus, while the cross section is Lorentz invariant, the chiral probabilities are not. This seems to suggest that probability measurements are chiral independent. We seem to have an argument against the physical significance of chiral oscillations. The reply to this objection, based on Lorentz invariance, is simply that in any given Lorentz frame chiral oscillations are manifestly important because of the chiral projection form (V–A) of the charged weak currents. The chiral probability variations produced by Lorentz transformations (even if γ^5 commutes with the Lorentz generators) are automatically compensated by the wave function normalization conditions[4] and the Lorentz transformations of the intermediate vector bosons and other participating particles.

In addition, the above discussion introduces the primary tools for accurately deriving the expression for the neutrino spin-flipping in a magnetic field, which can be related to chiral oscillations in the limit of a massless particle (ultra-relativistic limit). By correctly distinguishing the concepts of helicity and chirality, we can determine the origin and the influence of chiral oscillations and spin-flipping in the complete flavor conversion formula. In some previous manuscripts [10, 12, 15] we have also confirmed that the *fermionic* character of the particles modify the standard oscillation probability, which had previously been obtained by implicitly assuming the *scalar* nature of the mass eigenstates. Strictly speaking, we have obtained the term of very high oscillation frequency, depending on the sum of energies in the new oscillation probability formula, which, in the case of Dirac wave packets, represents modifications that introduce correction factors which, under the current experimental point of view, are not effective in the UR limit of propagating neutrinos in vacuum, but which deserves, at least, a careful investigation for neutrinos moving in the background matter, where spin/chiral effects become more relevant [19–23]. The physical consequences in environments such as supernovae can be theoretically studied [24]. For instance, it was observed that neutrinos propagating in matter get an effective electromagnetic vertex that affects the flavor conversion process in a framework where preserving chirality can be established [25].

Finally, just as a remark about this connection with neutrino physics, it is also to be noted that in this kind of analysis we have to assume that neutrinos are Dirac particles, thus making the positive-chiral component sterile. If neutrinos are Majorana particles [19], they cannot have a magnetic moment, obviating the spin-flipping via magnetic field interactions but still allowing for the (vacuum) chiral conversion possibility via very rapid oscillations (zitterbewegung).

To conclude, we have shown how chiral oscillations can be mathematically related to the zitterbewegung motion described by the velocity (Dirac) operator $\boldsymbol{\alpha}$. Once we have assumed that the neutrino electroweak interactions at the source and detector are the (*left*) chiral $\left(\overline{\psi}\gamma^{\mu}(1-\gamma^5)\psi W_{\mu}\right)$, only the component with negative chirality contributes to the propagation; therefore, chiral oscillations can take place in the context of neutrino quantum oscillation phenomena. We remind the reader that, in the usual (standard) treatment of vacuum neutrino oscillations, the use of scalar mass eigenstate wave packets composed exclusively of positive frequency plane wave solutions is implicitly assumed. Although the standard oscillation formula can predict the correct result when *properly* interpreted, a satisfactory description of fermionic (spin-half) particles requires the use of the Dirac equation as evolution equation for the propagating mass eigenstates. Consequently, the spinorial form and the interference between positive and negative frequency components of the mass eigenstate wave packets leads to the possibility of chirality coupled with flavor oscillations, which effectively introduces some small modifications to the *standard* flavor conversion formula [10] when it concerns non-relativistic neutrinos.

[4] In fact, the analytical form of localization is not frame independent.

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